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Minimum crossing numbers for 3-braids

Mitchell A Berger
Mathematics Department, University College London, Gower Street, London WC1E 6BT, UK

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Abstract. Given a braid on N strings, find an algorithm which generates an Artin braid word B of minimal length. This is an important unsolved problem—a solution would give us the most economical way of notating and drawing braids. The length of an Artin word equals the number of crossings seen in a braid diagram. Minimum crossing numbers provide a measure of complexity for braids. This paper presents an algorithm for $N = 3$. A three-dimensional configuration space for 3-braids will also be defined and analysed.

1. Introduction

A braid word B notes the successive crossings of a set of strings winding about each other between two parallel planes. We employ the Artin presentation for 3-braids

$$\{\sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}\,. \tag{1}$$

The length of a word is simply the number of characters: if $B = \sigma_1^{\alpha_1}\sigma_2^{\beta_1}\sigma_1^{\alpha_2}\dots$ then $L(B) = |\alpha_1| + |\beta_1| + |\alpha_2| \dots$. Many different braid words B can correspond to the same topological braid B. Figure 1 below shows a braid B, first represented by a word B_0 of length $L(B_0) = 12$, and then by an equivalent word $B_{\min}$ with the minimum possible length $L(B_{\min}) = 8$.

The diagrams in figure 1 project the braid onto a plane (say the x–z plane). The strings may be regarded as beginning and ending each crossing lined up parallel to the x-axis, deviating in the y-direction only to move around each other. However, other projections are possible, for example, onto a cylinder. Berger (1990a), in a study of braided magnetic flux tubes, treated the latter case. Minimal words were found, where the strings begin and end each crossing in an equilateral triangle. Independently, Pei Jun Xu (1993) found shortest words in the group of elementary 3-braids, which arises from cylindrical projections.

This paper presents an algorithm for the standard Artin braid group. In knot theory, the minimum crossing number provides a good measure of the complexity of knots and links (Soteros *et al* 1992); minimum word length provides a similar measure for braids (in fact, the word length for Artin braids equals the number of crossings in a braid diagram). Crossing numbers are especially useful in studying physical applications of knot and braid theory. Consider a set of flexible knotted or linked tubes, where we fill the interiors of the tubes with longitudinal magnetic fields. Freedman and He (1991) have shown that the total magnetic energy inside the tubes can be bounded below if we know the minimum crossing number (as averaged over all projection angles). Similarly, the energy of braided magnetic tubes can be estimated using minimum word length (Berger 1993). Such calculations are important for the study of braided magnetic fields in the atmosphere of the sun (Berger 1990b, 1994).

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The study of distributions of braids generated by random processes (Berger 1990a) will also clearly benefit from the existence of a measure of complexity. The algorithm for minimizing 3-braids will be described in terms of the Artin presentation; however, the proof of the algorithm relies on an analysis of the configuration space of geometric 3-braids. For a comprehensive introduction to braid theory, see Birman (1974).

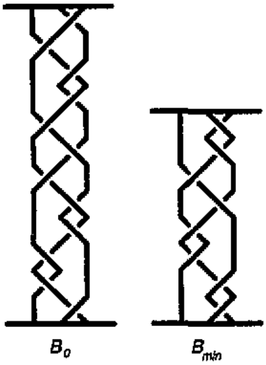

Figure 1. The two braids shown are equivalent. The braid $B_{\rm min}$ is obtained from B_0 by use of the algorithm.

2. The algorithm

Definition 2.1. $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

Definition 2.2. Given a braid word A, let $\hat{A}$ be the word constructed by replacing each σ_1 in A by σ_2, and vice versa.

For example, $A = \sigma_1\sigma_2^{-3}\sigma_1^{-1}$, $\hat{A} = \sigma_2\sigma_1^{-3}\sigma_2^{-1}$. Note that Δ corresponds to a uniform twist of the three strings through one half turn; it *almost* commutes with all other braid elements, in the sense that for any word A

$$A\Delta = \Delta\hat{A} \qquad A\Delta^{-1} = \Delta^{-1}\hat{A}\,. \tag{2}$$

One can show that Δ^2 generates the centre of the group of Artin words; it corresponds to a uniform twist of the three strings through one complete turn. The sequence Δ plays an important role in Garside's solution of the braid conjugacy problem (Arnol'd 1969).

Definition 2.3. A *wrap* is any one of the four sequences

$$\sigma_1\sigma_2 \qquad \sigma_2\sigma_1 \qquad \sigma_1^{-1}\sigma_2^{-1} \qquad \sigma_2^{-1}\sigma_1^{-1}\,. \tag{3}$$

In the following algorithm, we start with an arbitrary braid word B_0, which represents the braid B. We convert the braid into a form similar to the Schreier normal form (Birman 1974), and take care in placing twists.

Step (i). Search through B_0 for any occurence of Δ or Δ^{-1}. Whenever one is found, bring it to the left using $A\Delta = \Delta\hat{A}$. For example,

$$\sigma_1\Delta^3\sigma_2^{-2}\Delta^{-1} \to \Delta^3\sigma_2\sigma_2^{-2}\Delta^{-1} \to \Delta^2\sigma_1^{-1}\,.$$

Proceed until B_0 has been converted to a word of the form $\Delta^n B_1$ where B_1 is free of Δs.

Step (ii). Next clear away the wraps. By the fundamental relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$,

$$\begin{aligned} \sigma_1\sigma_2 &= \Delta\sigma_1^{-1} & \sigma_2\sigma_1 &= \Delta\sigma_2^{-1} \\ \sigma_1^{-1}\sigma_2^{-1} &= \Delta^{-1}\sigma_1 & \sigma_2^{-1}\sigma_1^{-1} &= \Delta^{-1}\sigma_2 \,. \end{aligned} \tag{4}$$

Search through B_1 for all wraps, and remove (unwrap) them by, for example, replacing $\sigma_1\sigma_2$ by $\Delta\sigma_1^{-1}$. Then bring the new Δs to the front as in step (i). The result is either $B = \Delta^p B_2$ or $B = \Delta^p \hat{B}_2$, where

$$B_2 = \sigma_1^{q_1}\sigma_2^{-r_1}\sigma_1^{q_2}\sigma_2^{-r_2}\ldots\sigma_1^{q_m}\sigma_2^{-r_m} \,. \tag{5}$$

Here the qs and rs are positive integers. Let $q = \sum_{i=1}^{m} q_i$ and $r = \sum_{i=1}^{m} r_i$. If $\mathcal{W}(B)$ is the writhe of B (algebraic sum of exponents of the σs) then $\mathcal{W}(B) = 3p + q - r$.

Step (iii). Here we partially reverse step (ii). For definiteness, suppose that $p > 0$ and that $B = \Delta^p B_2$. One at a time, we bring a Δ to the right and combine it with the first σ_2^{-1} remaining in the word to form the wrap $\sigma_2\sigma_1$. Each time this is done, length is reduced by 2. Thus,

$$\begin{aligned} B &= \Delta^p\sigma_1^{q_1}\sigma_2^{-r_1}\ldots\sigma_1^{q_m}\sigma_2^{-r_m} \\ &= \Delta^{p-1}\sigma_2^{q_1}(\Delta\sigma_2^{-1})\sigma_2^{-r_1+1}\ldots\sigma_1^{q_m}\sigma_2^{-r_m} \\ &= \Delta^{p-1}\sigma_2^{q_1}(\sigma_2\sigma_1)\sigma_2^{-r_1+1}\ldots\sigma_1^{q_m}\sigma_2^{-r_m} \,. \end{aligned}$$

Now repeat until either there are no Δs remaining ($p \leqslant r$) or no σ_2^{-1}s remaining ($p \geqslant r$). The final result will be called $B_{\min}$.

Theorem 2.1. Given B, the word $\Delta^p B_2$ is unique. The word $B_{\min}$ has the minimum length L of any word for B, and is uniquely specified.

Proof will be given in section 4. Here we give a short intuitive explanation of the algorithm, and an example (figure 1). In step (i), length L is reduced by six each time there is a cancellation between a Δ and a Δ^{-1}. There may be other cancellations, as in the example for step (i). In step (ii), L increases by two for each unwrapping, but there may be more $\Delta\Delta^{-1}$ cancellations. Thus if at one place in the word $\sigma_1\sigma_2$ is unwrapped and at another place $\sigma_1^{-1}\sigma_2^{-1}$ is unwrapped, there is a net length decrease of 2 after cancellation. If there are still Δs left over after step (ii), then it is most efficient to put them back into wraps as done in step (iii). Figure 1 illustrates the following example:

$$\begin{aligned} B_0 &= \sigma_2\sigma_1^{-2}\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{2}\sigma_1\sigma_2 \\ &= \sigma_2\sigma_1^{-2}\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\Delta \\ &= \Delta\sigma_1\sigma_2^{-2}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1 && = \Delta B_1 \\ &= \Delta\sigma_1\sigma_2^{-1}(\sigma_2^{-1}\sigma_1^{-1})\sigma_1^{-1}\sigma_2\sigma_1^{-1}(\sigma_2\sigma_1) \\ &= \Delta\sigma_1\sigma_2^{-1}(\Delta^{-1}\sigma_2)\sigma_1^{-1}\sigma_2\sigma_1^{-1}(\Delta\sigma_2^{-1}) \\ &= \Delta\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2} && = \Delta B_2 \\ &= \sigma_2(\Delta\sigma_2^{-1})\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2} \\ &= \sigma_2^{2}\sigma_1^{2}\sigma_2^{-1}\sigma_1\sigma_2^{-2} && = B_{\min} \,. \end{aligned}$$

We note here that open braids can be minimized in a similar manner. For an open braid top and bottom planes are identified. Thus we can cut the braid in two and switch the positions of the pieces: if $B = B_a B_b$, then by bringing the top piece of the braid B_b through the top/bottom plane, we get $B = B_b B_a$. Suppose we minimize the braid through step (ii) so that $B = \Delta^p B_2$, with B_2 as in (5). First suppose that p is even. We can apply step (iii) to get $B_{\rm min}$ as usual. Alternatively, we can bring a piece of the braid through the top/bottom plane first. For example, if $r_m \neq 0$

$$\begin{aligned} B &= \Delta^p \sigma_1^{q_1} \ldots \sigma_1^{q_m} \sigma_2^{-r_m} = \sigma_2^{-r_m} \Delta^p \sigma_1^{q_1} \ldots \sigma_1^{q_m} \\ &= \Delta^p \sigma_2^{-r_m} \sigma_1^{q_1} \ldots \sigma_1^{q_m}. \end{aligned}$$

The normal form has changed, but otherwise there has been no change in length. Application of step (iii) results in a different $B_{\rm min}$ but with the same length.

If p is odd, on the other hand, the braid may minimize further. Let us move σs through one at a time. If the last σ is different from the first, then there will be a cancellation:

$$\begin{aligned} B &= \Delta^p \sigma_1^{q_1} \ldots \sigma_2^{-r_m} = \sigma_2^{-1} \Delta^p \sigma_1^{q_1} \ldots \sigma_2^{-r_m+1} \\ &= \Delta^p \sigma_1^{-1} \sigma_1^{q_1} \ldots \sigma_2^{-r_m+1}. \end{aligned}$$

Eventually the last σ will be of the same type as the first. In this case bringing the last one through will create a wrap, for example,

$$\begin{aligned} B &= \Delta^p \sigma_1^{q_1} \ldots \sigma_1^{q_m} = \sigma_1 \Delta^p \sigma_1^{q_1} \ldots \sigma_1^{q_m-1} \\ &= \Delta^p (\sigma_2 \sigma_1) \sigma_1^{q_1-1} \ldots \sigma_1^{q_m-1} = \Delta^{p+1} \sigma_2^{-1} \sigma_1^{q_1-1} \ldots \sigma_1^{q_m-1}. \end{aligned}$$

Now Δ has an even exponent, and step (iii) can be applied to obtain a minimized braid.

3. The configuration space for 3-braids

This section explores the geometry of 3-braids, using the winding number techniques introduced in Berger (1991). Geometric braids are discussed in general terms in chapter 1.1 of Birman (1974).

Definition 3.1. Let $F_{0,3}\mathbb{C}$ be the space

$$F_{0,3}\mathbb{C} = \{(a, b, c) | a, b, c \in \mathbb{C}, a \neq b, b \neq c, c \neq a\}. \tag{6}$$

Definition 3.2. A geometric braid is a curve $\gamma : [0, 1] \to F_{0,3}\mathbb{C}$. We can regard γ as recording the history of three points $a(t)$, $b(t)$, and $c(t)$ moving in the complex plane between times $t = 0$ and $t = 1$. Two geometric braids γ_1 and γ_2 are considered equivalent, $\gamma_1 \sim \gamma_2$, if there is an isotopic deformation of $\mathbb{C} \times [0, 1]$ which is the identity on $\mathbb{C} \times \{0\}$ and $\mathbb{C} \times \{1\}$, and which sends γ_1 to γ_2. It will be convenient to make the following choice for initial positions of a, b, and c:

$$\gamma(0) = (a, b, c)(0) = (0, 1, 2). \tag{7}$$

Also we will suppose that $\gamma(1)$ consists of a permutation of the points 0, 1, 2.

Definition 3.3. Define one-forms ω_{ab}, ω_{bc}, ω_{ca}, and their integrals λ_{ab}, λ_{bc}, λ_{ca} by, e.g.,

$$\omega_{ab} = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}b - \mathrm{d}a}{b - a} \tag{8a}$$

$$\lambda_{ab}(\gamma(t)) = \int_{\gamma(0)}^{\gamma(t)} \omega_{ab} \,. \tag{8b}$$

We can write

$$\lambda_{ab}(\gamma(t)) = \lambda_{ba}(\gamma(t)) = \frac{1}{2\pi \mathrm{i}} \log \frac{b(t) - a(t)}{b(0) - a(0)} \tag{9}$$

with the understanding that λ_{ab} takes its values on a Reimann surface above the complex plane.

Definition 3.4. Define three winding numbers ψ_{ab}, ψ_{bc}, and ψ_{ca} by, e.g.,

$$\begin{aligned} \psi_{ab}(t) &= \mathrm{Re}\,\lambda_{ab}(\gamma(t)) \\ &= \frac{1}{2\pi} \mathrm{Im} \log \frac{b(t) - a(t)}{b(0) - a(0)} \,. \end{aligned} \tag{10}$$

This measures the net winding of string a about string b between 0 and t in units of complete turns. (Berger (1991) employs the winding *angle* $\theta_{ab} = 2\pi \psi_{ab}$). Also let the total winding number be

$$W(t) = \psi_{ab}(t) + \psi_{bc}(t) + \psi_{ca}(t) \,. \tag{11}$$

Definition 3.5. Let $\mathcal{T}_3 \subset \mathbb{R}_3$ be the set of all admissible triples $(\psi_{ab}, \psi_{bc}, \psi_{ca})$. Admissible means that a triangle in the complex (or Euclidean) plane can have these winding numbers.

For a right-handed non-degenerate triangle (the path $a \to b \to c$ goes anti-clockwise) an admissible triple $(\psi_{ab}, \psi_{bc}, \psi_{ca})$ must satisfy

$$\begin{aligned} &0 < (\psi_{ca} - \psi_{ab}) \bmod 1 < \tfrac{1}{2} \\ &0 < (\psi_{bc} - \psi_{ca}) \bmod 1 < \tfrac{1}{2} \\ &0 < -(\psi_{ab} - \psi_{bc}) \bmod 1 < \tfrac{1}{2} \,. \end{aligned} \tag{12}$$

For left-handed triangles

$$\begin{aligned} &0 < (\psi_{ab} - \psi_{ca}) \bmod 1 < \tfrac{1}{2} \\ &0 < (\psi_{ca} - \psi_{bc}) \bmod 1 < \tfrac{1}{2} \\ &0 < -(\psi_{bc} - \psi_{ab}) \bmod 1 < \tfrac{1}{2} \,. \end{aligned} \tag{13}$$

Definition 3.6. A connected region in $\mathcal{T}_3$ satisfying (12) will be called a *right-handed prism*. A connected region in $\mathcal{T}_3$ satisfying (13) will be called a *left-handed prism*.

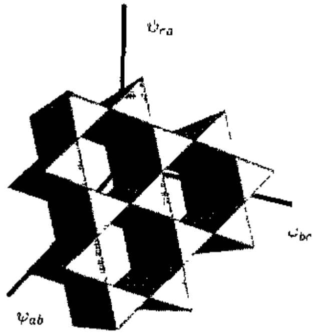

Figure 2. The phase space $\mathcal{T}_3$. The hexagonal holes are forbidden regions.

Note that if $(\psi_{ab}, \psi_{bc}, \psi_{ca})$ is admissible, then so is $(\psi_{ab}+x, \psi_{bc}+x, \psi_{ca}+x)$ where x is any real number. Thus the geometry of $\mathcal{T}_3$ is non-trivial only in a plane perpendicular to the direction $(1, 1, 1)$. One such plane is the plane $W = 0$.

Definition 3.7. Let $\mathcal{P}_3 \subset \mathcal{T}_3$ be the plane $\{(\psi_{ab}, \psi_{bc}, \psi_{ca}) | \, \psi_{ab} + \psi_{bc} + \psi_{ca} = 0\}$.

See figure 2. The triangular regions correspond to slices through prisms. The edges of these regions (faces of the prisms) are forbidden. The vertices (edges of the prisms) correspond to degenerate triangles, i.e. where the three points a, b, and c are colinear. The centre of a triangular region in $\mathcal{P}_3$ corresponds to a, b, and c forming an equilateral triangle. Define coordinates $\phi_{ab}, \phi_{bc}, \phi_{ca}$ by $\phi_{ij} = 6\psi_{ij} - 2W$. The factor of 6 makes ϕ_{ij} an integer at the vertices. We can then project $\mathcal{T}_3$ onto $\mathcal{P}_3$ by

$$(\psi_{ab}, \psi_{bc}, \psi_{ca}) \to (\phi_{ab}, \phi_{bc}, \phi_{ca}) = 6(\psi_{ab}, \psi_{bc}, \psi_{ca}) - 2(W, W, W) . \tag{14}$$

The net winding numbers $\Psi_{ij} = \psi_{ij}(1)$ are invariants to isotopic deformations, i.e. if $\gamma_1 \sim \gamma_2$ then $\Psi_{1ab} = \Psi_{2ab}$. To see this, note that Ψ_{ab} is obtained by integration of the form ω_{ab}; because ω_{ab} is closed this integral is the same for homotopic paths γ_1 and γ_2.

Higher-order winding numbers for braids can be defined in a similar manner (Berger 1991, Evans and Berger 1992). For example, the third-order invariant is

$$\Psi_{abc} = \int_{\gamma(0)}^{\gamma(1)} [\lambda_{ab}\omega_{bc} + \lambda_{bc}\omega_{ca} + \lambda_{ca}\omega_{ab} - \lambda_{ab}\omega_{ca} - \lambda_{bc}\omega_{ab} - \lambda_{ca}\omega_{bc}] . \tag{15}$$

These invariants (for closed braids with vanishing lower-order invariants) correspond to Massey higher-order linking numbers and Milnor $\bar{\mu}$ numbers.

The curve $\gamma \in \mathcal{C}_3$ induces new curves $\bar{\gamma} \in \mathcal{T}_3$ and $\tilde{\gamma} \in \mathcal{P}_3$. These will be called *phase curves* as they describe the evolution of the angles between the three points a, b, and c. Two phase curves $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are equivalent if one can be obtained from the other by an isotopic deformation. From the preceding discussion $\bar{\gamma}(1) = (\Psi_{ab}, \Psi_{bc}, \Psi_{ca})$ is invariant to isotopic deformations. Also the beginning point $\bar{\gamma}(0) = (0, 0, 0)$ is fixed.

Theorem 3.1. Two geometric braids are equivalent, $\gamma_1 \sim \gamma_2$, if and only if $\bar{\gamma}_1 \sim \bar{\gamma}_2$.

Proof. If $\gamma_1 \sim \gamma_2$ then there exists a homotopy $\Gamma(t, s)$ where $\Gamma(t, 0) = \gamma_1(t)$, $\Gamma(t, 1) = \gamma_2(t)$, and for fixed s_0, $\Gamma(t, s_0)$ is a geometric braid. Similarly, $\bar{\gamma}_1 \sim \bar{\gamma}_2$ implies the existence of $\bar{\Gamma}(t, s)$ where $\bar{\Gamma}(t, 0) = \bar{\gamma}_1(t)$, $\Gamma(t, 1) = \bar{\gamma}_2(t)$, and $\bar{\Gamma}(t, s_0)$ is a path through $\mathcal{T}_3$. Given a curve $\gamma(t)$, $\bar{\gamma}(t) = G(\gamma(t))$ where $G : F_{0,3}\mathbb{C} \to \mathcal{T}_3$ is given by (8) and (10). Suppose $\gamma_1 \sim \gamma_2$. Then let $\bar{\Gamma}(t, s) = G(\Gamma(t, s))$ for any fixed s. This is the required homotopy giving $\bar{\gamma}_1 \sim \bar{\gamma}_2$. Next suppose $\bar{\gamma}_1 \sim \bar{\gamma}_2$. Then there exists a homotopy

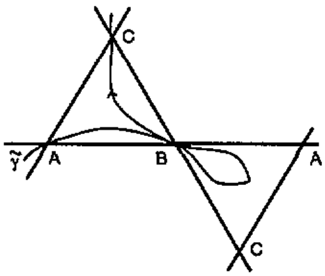

Figure 3. The phase curve $\tilde{\gamma}$ can be deformed in order to remove the loop through the vertex B. The word ABBC is equivalent to AC.

$\bar{\Gamma}(t,s) = (\psi_{ab}, \psi_{bc}, \psi_{ca})(t,s)$. Given $\gamma_1(t)$ and $\gamma_2(t)$, we construct the homotopy between them $\Gamma(t,s) = (a,b,c)(t,s)$ as follows: first,

$$a(t,s) = (1-s)a_1(t) + s\,a_2(t)\,. \tag{16}$$

Secondly, let

$$r_{ab}(t,s) = (1-s)|b_1(t) - a_1(t)| + s|b_2(t) - a_2(t)| \tag{17}$$

then

$$b(t,s) = a(t,s) + r_{ab}(t,s)\mathrm{e}^{\mathrm{i}\psi_{ab}(t,s)} \tag{18}$$

and finally

$$c(t,s) = a(t,s) + r_{ca}(t,s)\mathrm{e}^{\mathrm{i}\psi_{ca}(t,s)} \tag{19}$$

where (for $\sin(\psi_{ab}(t,s) - \psi_{bc}(t,s) \neq 0)$

$$\frac{r_{ca}(t,s)}{\sin(\psi_{ab}(t,s) - \psi_{bc}(t,s))} = \frac{r_{ab}(t,s)}{\sin(\psi_{ab}(t,s) - \psi_{bc}(t,s))}\,. \tag{20}$$

For isolated values of (t,s) where there is a degeneracy, $\sin(\psi_{ab}(t,s) - \psi_{bc}(t,s)) = 0$, we can choose $r_{ca}(t,s)$ by continuity. Degeneracies can always be isolated, if necessary, by a slight deformation of $\bar{\Gamma}(t,s)$.

Theorem 3.2. An equivalence class of geometric braids is completely and uniquely specified by (i) the writhe $\mathcal{W} = 2W(1) = 2(\Psi_{ab} + \Psi_{bc} + \Psi_{ca})$, (ii) a word P formed from three letters A, B, and C, where no two consecutive letters are identical, and (iii) a sign $s \in \{-1, 1\}$.

Proof. Since a phase curve $\tilde{\gamma}$ can be deformed without obstruction in the direction (1,1,1) of increasing W, we can always set

$$W(t) = tW(1)\,. \tag{21}$$

In this case, two equivalent geometric braids satisfying (21) will only differ in their projection $\tilde{\gamma}$ in $\mathcal{P}_3$. We can notate $\tilde{\gamma}$ by recording the sequence of vertices in $\mathcal{P}_3$ that $\tilde{\gamma}$ passes through. Recall that the points a, b, c are colinear when $\tilde{\gamma}$ passes through a vertex. Label a vertex 'A' if the point a is in the middle, 'B' for b in the middle, and 'C' for c in the middle. Starting at $t = 0$, write down B for the initial point (chosen for definiteness to be $(a,b,c) = (0,1,2)$) and write a label whenever $\tilde{\gamma}$ passes transversely *through* a vertex. End the word with the label of the final point.

A deformation of a section of $\tilde{\gamma}$ can only change the word if that section loops through a vertex, as in figure 3. Thus the word can only change by insertion or deletion of repeated letters. Conversely, if the word contains repeated letters, then these can be removed by isotopic deformation. This task can be performed until there are no remaining repeats. (For example, BACABBABC $\rightarrow$ BACAABC $\rightarrow$ BACBC.) The curve $\tilde{\gamma}$ passes alternately through left and right-handed prisms. The sign s specifies the handedness of the first prism that $\tilde{\gamma}$ passes through. For example if P = BACBC then there are two choices for the A vertex (σ_1 passing through a left-handed prism ($s = -1$), and σ_1^{-1} passing through a right-handed prism ($s = 1$).

4. The phase curves of Artin braids

What do Artin words generated from σ_1 and σ_2 look like when represented by $\bar{\gamma}$ or $\tilde{\gamma}$? Each σ in the Artin word corresponds to a transition of the geometrical braid: the three points a, b, and c begin and end on the real line, but their ordering is permuted. The section of $\bar{\gamma}$ corresponding to a σ begins and ends on the edge of a prism; similarly, the section of $\tilde{\gamma}$ begins at a vertex in $\mathcal{P}_3$, passes through one triangular region, and ends at a new vertex. Also, each σ changes W by $\pm\frac{1}{2}$. The identity $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is illustrated in figure 4. Both these words correspond to the half twist Δ; their phase curves $\bar{\gamma}$ can be deformed into a vertical path (along a prism edge) in the direction of increasing W, but with projection $\tilde{\gamma}$ staying at a single point.

Figures 5 and 6 show σ_1 and the wrap $\sigma_1^{-1}\sigma_2^{-1}$, starting with the initial points $(0, 1, 2)$. Both make a transition from vertex B to vertex A (the wrap touches the vertex C but does not pass through). Starting from any vertex, there are four neighbouring vertices. These can be reached directly, by the four moves σ_1, σ_2, σ_1^{-1}, σ_2^{-1}, or indirectly by adding powers of Δ, for example to form wraps. It will be useful here to factor out the centre of the group of (isotopically equivalent) phase curves $\bar{\gamma}$.

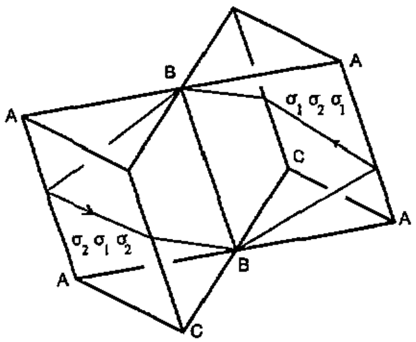

Figure 4. The phase curves $\bar{\gamma}$ for the equivalent braids $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$.

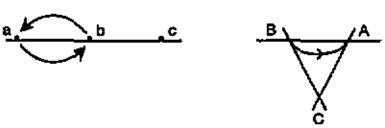

Figure 5. The braid element σ_1 and its phase curve $\tilde{\gamma}$.

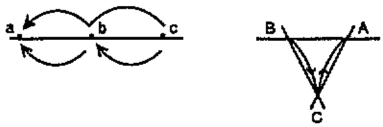

Figure 6. The wrap $\sigma_1^{-1}\sigma_2^{-1}$ and its phase curve $\tilde{\gamma}$.

Definition 4.1. Given two phase curves $\bar{\gamma}_1$ and $\bar{\gamma}_2$, $\bar{\gamma}_1 \simeq \bar{\gamma}_2$ if $\tilde{\gamma}_1 \sim \tilde{\gamma}_2$ (i.e. if $\tilde{\gamma}_1$ can be deformed into $\tilde{\gamma}_2$).

Since $\Delta \simeq 0$ (a curve consisting of a single point), by (4) $\bar{\gamma}(\sigma_1^{-1}\sigma_2^{-1}) \simeq \bar{\gamma}(\sigma_1)$ as shown in figures 5 and 6.

Proof of theorem 2.1. Let a braid B have phase curve $\bar{\gamma}$ and let $[\bar{\gamma}]$ be its equivalence class under $\simeq$. Each member of $[\bar{\gamma}]$ has the same minimal vertex sequence P. A transition between vertices in P can have word length $1, 2, 4, 5, 7, 8, \ldots$ (e.g. σ_1, $\sigma_1^{-1}\sigma_2^{-1}$, $\Delta\sigma_1$, $\Delta^{-1}\sigma_1^{-1}, \ldots$). Given each transition there is exactly one way of making the transition using a single σ. Then there is a unique curve $\bar{\gamma}_2$ with Artin word B_2 formed by making each transition with a single σ. Let $L(P)$ = (number of letters in P) $- 1$. Then $L(B_2) = L(P)$. If $\bar{\gamma} \not\simeq \bar{\gamma}_2$ then $\bar{\gamma}$ and $\bar{\gamma}_2$ differ by a power p of Δ. Since the writhe is an invariant, $p = (\mathcal{W}(\bar{\gamma}) - \mathcal{W}(\bar{\gamma}_2))/3$. The factor Δ^p can be placed in front to make $B = \Delta^p B_2$. The length of the word may vary depending on where the Δs are placed, or in other words, where $\bar{\gamma}$ climbs above or below $\bar{\gamma}_2$. The net length of $\Delta^p B_2$ can be decreased only by forming wraps from a Δ and a σ^{-1} or from a Δ^{-1} and a σ. Thus if p is positive, the Δs should be distributed among the σ^{-1}s. If $p > r$ there will be $p!/r!(p-r)!$ ways of doing this. A unique distribution can be specified by converting the first p σ^{-1}s into wraps.

We conclude by considering the general problem of minimizing N-braids for $N > 3$. We can compute the dimension of configuration space for N-braids as follows: at each value of t we can translate the braid so that the first string stays at $(x, y) = (0, 0)$. This leaves $2(N-1)$ dimensions. Scaling and projecting out twist removes 2 more dimensions. Thus the dimension of the space analogous to $\mathcal{P}_3$ is $2(N-2)$. Only for $N = 3$ is this small enough to make the problem simple.

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